On Differential Forms

Abstract. This article will give a very simple definition of k-forms or differential forms. It just requires basic knowledge about matrices and determinants. Furthermore a very simple proof will be given for the proposition that the double outer differentiation of k-forms vanishes.

MSC 2010: 58A10

1. Basic definitions.

We denote the submatrix of $A = (a_{ij}) \in R^{m\times n}$ consisting of the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$ with

$$[A]^{j_1 \ldots j_k}_{i_1 \ldots i_k} := \begin{pmatrix} a_{i_1 j_1} & \ldots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \ldots & a_{i_k j_k} \end{pmatrix}$$

and its determinant with

$$A^{j_1 \ldots j_k}_{i_1 \ldots i_k} := \det[A]^{j_1 \ldots j_k}_{i_1 \ldots i_k}.$$

For example

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \qquad A^{1,3}_{1,2} = a_{11}a_{23} - a_{21}a_{13}.$$

Suppose

$$H \in R^{n\times(n+1)}$$

and let

$$f, g \colon U \subseteq R^n \to R, \qquad U \text{ open}$$

be two functions which are two-times continuously differentiable. Then we call for a fixed k the expression

$$f\, H^{1 \ldots k}_{\alpha}, \qquad \alpha = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k,$$

a *basic k-form* or *basic differential form* of order k. It's a real function of $n + k^2$ variables. For $k > n$ the expression is defined to be zero. If f also depends on α then

$$\sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \ldots i_k} H^{1\ \ldots\ k}_{i_1 \ldots i_k}$$

is called a *k-form.* It's a real function of $n + kn$ variables which is k-linear in the k column-vectors of H.

For example for $f \colon R \to R$ and $H \in R^{1\times 1}$ we have $f(x)\, H$. This is a linear function in H and a possibly non-linear function in x.

2. Differentiation of k-forms.

For the differential form

$$\omega = f H_{\alpha}^{1\ldots k}, \qquad \alpha = (i_1, \ldots, i_k)\,,$$

we define

$$d\omega := \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} H_{\nu,\alpha}^{1\ldots k+1}$$

as the *outer differentiation* of ω. This is a $(k+1)$-form. It's a function of $n + (k+1)n$ variables.

The 0-form

$$\omega = f, \qquad |\alpha| = k = 0$$

yields

$$dw = \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} H_{\nu}^{1} \tag{1}$$

which corresponds to $\nabla f = \operatorname{grad} f$.

In the special case $k = |\alpha| = 1$ we get for

$$\omega = \sum_{i=1}^{n} f_i H_i^1$$

the result

$$d\omega = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} H_{j,i}^{1,2} = \sum_{i<j} \left(\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} \right) H_{j,i}^{1,2}. \tag{2}$$

This corresponds to $\operatorname{rot} f$.

Let hat $\hat{()}$ mean exclusion from the index list. The case $k = n-1$ for

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} f_i \, H_{1\ldots\hat{\imath}\ldots n}^{1\ldots n-1}$$

delivers

$$dw = \sum_{i=1}^{n} \sum_{\nu=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_\nu} H_{\nu,1\ldots\hat{\imath}\ldots n}^{1\ldots n} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_\nu} H_{1\ldots n}^{1\ldots n} = \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right) \det H.$$

This corresponds to $\operatorname{div} f$.

Theorem. *For $\omega = f H_{\alpha}^{1\ldots k}$ we have*

$$dd\omega = 0.$$

Proof: With

$$d\omega = \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} H_{\nu,\alpha}^{1\ldots k+1}$$

we get

$$dd\omega = \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} \frac{\partial^2 f}{\partial x_\nu \partial x_\mu} H_{\mu,\nu,\alpha}^{1\ldots k+2}$$

and this is zero, because

$$H_{\mu,\mu,\alpha}^{1\ldots k+2} = 0, \qquad H_{\mu,\nu,\alpha}^{1\ldots k+2} = -H_{\nu,\mu,\alpha}^{1\ldots k+2},$$

and

$$\frac{\partial^2 f}{\partial x_\nu \partial x_\mu} = \frac{\partial^2 f}{\partial x_\mu \partial x_\nu}.$$

Application of this theorem to an 0-form with an $f\colon U \subseteq R^n \to R$ and a 1-form with an $a\colon U \to R^n$ reading (1) and then (2) yields

$$\operatorname{rot}\operatorname{grad} f = 0, \qquad \operatorname{div}\operatorname{rot} a = 0.$$

The second equation is only true for $n = 3$ because

$$\binom{n}{2} = n \quad (n \in N) \qquad \Leftrightarrow \qquad n = 3.$$

Definition. Suppose

$$\phi\colon D \to E \subset R^n, \qquad D \subset\subset R^k,$$

is differentiable, its derivative denoted by ϕ', and

$$f\colon E \to R.$$

For the differential form $\omega = fH_\alpha^{1\ldots k}$ we define the *back-transportation* as

$$\phi^*\omega := (f \circ \phi)\,(\phi')_\alpha^{1\ldots k}$$

and the integral over k-forms as

$$\int_\phi \omega := \int_D \phi^*\omega.$$

For example the case $k = 1$,

$$\omega = \sum_{i=1}^n f_i H_i^1$$

gives

$$\phi^*\omega = \sum_{i=1}^n (f_i \circ \phi)\,(\phi')_i^1.$$

3. The outer product of differential forms.

Suppose

$$H \in R^{n\times(n+1)}, \qquad k+m \le n.$$

For the two differential forms

$$\omega = \sum_{1\le i_1<\cdots<i_k\le n} f_{i_1\ldots i_k} H^{1\ \ldots\ k}_{i_1\ldots i_k}$$

and

$$\lambda = \sum_{1\le j_1<\cdots<j_m\le n} g_{j_1\ldots j_m} H^{k+1\ldots k+m}_{j_1\ \ldots\ j_m}$$

the outer product is defined as

$$w\wedge\lambda := \sum_{\substack{1\le i_1<\cdots<i_k\le n\\ 1\le j_1<\cdots<j_m\le n}} f_{i_1\ldots i_k} g_{j_1\ldots j_m} H^{1\ \ldots\ k\ k+1\ldots k+m}_{i_1\ldots i_k\ j_1\ \ldots\ j_m}.$$

This is a differential form of order $k+m$. It's a function in $n+(k+m)n$ variables.

Theorem.

$$d(\omega\wedge\lambda) = d\omega\wedge\lambda + (-1)^k\omega\wedge d\lambda$$

Proof: With

$$\omega = \sum_\alpha f_\alpha H^{1\ldots k}_\alpha, \qquad \lambda = \sum_\beta g_\beta H^{1\ldots m}_\beta$$

then

$$\begin{aligned} d(\omega\wedge\lambda) &= \sum_{\alpha,\beta}\sum_{\nu=1}^n \left(\frac{\partial f_\alpha}{\partial x_\nu} g_\beta + f_\beta \frac{\partial g_\beta}{\partial x_\nu}\right) H^{1\ldots k+m+1}_{\nu,\alpha,\beta} \\ &= \sum_{\alpha,\beta}\sum_{\nu=1}^n \frac{\partial f_\alpha}{\partial x_\nu} g_\beta H^{1\ldots k+m+1}_{\nu,\alpha,\beta} + \sum_{\alpha,\beta}\sum_{\nu=1}^n f_\alpha \frac{\partial g_\beta}{\partial x_\nu} H^{1\ldots k+m+1}_{\nu,\alpha,\beta} \\ &= d\omega\wedge\lambda + (-1)^k\omega\wedge d\lambda, \end{aligned}$$

due to

$$H^{1\ldots k+m+1}_{\nu,\alpha,\beta} = (-1)^k H^{1\ldots k+m+1}_{\nu,\beta,\alpha}$$

and

$$d\lambda = \sum_\beta \sum_{\nu=1}^n \frac{\partial g_\beta}{\partial x_\nu} H^{1\ldots m+1}_{\nu,\beta}.$$

An alternative definition for the differentiation of k-forms could be given.

Theorem. *Suppose*

$$\omega = f H^{1\ldots k}_\alpha, \qquad 0\le|\alpha|\le k,$$

and

$$H = (h_1,\ldots,h_n,h_{n+1}) \in R^{n\times(n+1)}$$

with $\alpha = (i_1,\ldots,i_k)$ *we have*

$$d\omega = \det\left(\mathrm{col}\left(\nabla f, [\mathrm{Id}_n]^{1\ldots n}_\alpha\right)[H]^{1\ldots k+1}_{1\ldots n}\right) = \sum_{\nu=1}^n \frac{\partial f}{\partial x_\nu} H^{1\ldots k+1}_{\nu,\alpha},$$

where col *just stacks matrices one above another and* Id_n *is the identity matrix in* R^n*.*

Proof:

$$d\omega = \begin{vmatrix} \langle \nabla f, h_1 \rangle & \dots & \langle \nabla f, h_k \rangle & \langle \nabla f, h_{k+1} \rangle \\ \langle e_{i_1}, h_1 \rangle & \dots & \langle e_{i_1}, h_k \rangle & \langle e_{i_1}, h_{k+1} \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle e_{i_k}, h_1 \rangle & \dots & \langle e_{i_k}, h_k \rangle & \langle e_{i_k}, h_{k+1} \rangle \end{vmatrix}$$

$$= \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} \begin{vmatrix} h_{1,\nu} & h_{1,i_1} & \dots & h_{1,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k,\nu} & h_{k,i_1} & \dots & h_{k,i_k} \\ h_{k+1,\nu} & h_{k+1,i_1} & \dots & h_{k+1,i_k} \end{vmatrix}$$

since

$$\langle \nabla f, h_1 \rangle = \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} h_{1,\nu},$$

$$\vdots \qquad \vdots$$

$$\langle \nabla f, h_{k+1} \rangle = \sum_{\nu=1}^{n} \frac{\partial f}{\partial x_\nu} h_{k+1,\nu}.$$

REFERENCES.

1. Walter Rudin, Principles of Mathematical Analysis, Second Edition, McGraw-Hill, New York, 1964

2. Otto Forster, Analysis 3: Integralrechnung im R^n mit Anwendungen, Third Edition, Friedrich Vieweg & Sohn, Braunschweig/Wiesbaden, 1984

Author's address:

Elmar Klausmeier
Goethestrasse 4
D-63128 Dietzenbach
Germany
`http://eklausmeier.wordpress.com`